A mean-field theory of the elastic granular disc model in two dimensions

1989 J. Phys. A: Math. Gen. 22 L401

(http://iopscience.iop.org/0305-4470/22/9/009)

J. Phys. A: Math. Gen. 22 (1989) L401–L406. Printed in the UK

LETTER TO THE EDITOR

A mean-field theory of the elastic granular disc model in two dimensions

Jian Wang

Department of Chemistry, Brandeis University, Waltham, MA 02254, USA

Received 20 January 1989

Abstract. We have constructed a Stephen-type mean-field theory for the elastic granular disc model in two dimensions. We have calculated the mean-field value of two crossover exponents ϕ_1 and ϕ_2, which describe the ways in which the two-point elastic susceptibility and the angle–angle correlation function scale with the distance, respectively. We found that $\phi_1 = 2$ and $\phi_2 = 1$. The possible relation of ϕ_1 and ϕ_2 to the bulk modulus exponent f is discussed.

Recently much attention has been directed towards randomly diluted elastic networks. Most studies have focused on the following aspects: (1) computer simulation [1–5]; (2) scaling analysis [6, 7]. Up to now, a field theory or even a mean-field theory to understand the critical behaviour of an elastic network has not been proposed. In this letter, we give a Stephen-type mean-field [8] theory for a two-dimensional granular disc model introduced by Feng [9]. We introduce two crossover exponents ϕ_1 and ϕ_2 describing the ways in which the two-point elastic susceptibility and splay elastic susceptibility (or angle–angle correlation function) scale with the distance, respectively. We show that as we turn on the angular part in the granular disc Hamiltonian, we have a crossover from $\phi_1 = 1$ to $\phi_1 = 2$. Using a node link picture [10], one can get a scaling relation relating ϕ_1 to the bulk modulus exponent f as

$$f = (d-2)\nu + \phi_1. \qquad (1)$$

Our result gives $f_{\mathrm{MF}} = 4$ which agrees with the scaling analysis [6]. A possible relation between ϕ_2 and ϕ_1 would be $\phi_1 = \phi_2 + 2\nu$ from dimensional analysis. Hence we may have

$$f = d\nu + \phi_2. \qquad (2)$$

The Hamiltonian of the granular disc model [9] can be written as follows:

$$H = \sum_{\langle ij\rangle} H_{ij}\varepsilon_{ij} \qquad (3)$$

where ε_{ij} is an indicator variable: $\varepsilon_{ij} = 1$ if bond b is occupied and $\varepsilon_{ij} = 0$ otherwise, and $\langle ij\rangle$ indicates a sum over pairs of nearest-neighbour sites, where

$$H_{ij} = \tfrac{1}{2}\alpha_1|(\boldsymbol{u}_i - \boldsymbol{u}_j)\cdot\hat{\boldsymbol{R}}_{ij}|^2 + \tfrac{1}{2}\beta_1|((\boldsymbol{u}_i - \boldsymbol{u}_j)\times\hat{\boldsymbol{R}}_{ij})\times\hat{\boldsymbol{R}}_{ij} + R(\theta_i + \theta_j)\hat{z}\times\hat{\boldsymbol{R}}_{ij}|^2 + \tfrac{1}{2}\gamma_1 R^2|\theta_i - \theta_j|^2 \qquad (4)$$

where $\boldsymbol{u}_i$ is the displacement associated with site i, $\hat{\boldsymbol{R}}_{ij}$ is a unit vector along sites i and j, θ_i is the angle of the disc at site i and $\hat{z}$ is a unit vector perpendicular to the two-dimensional system. Here α_1, β_1 and γ_1 are elastic constants; R is the radius of

0305-4470/89/090401+06$02.50

the disc. Note that this model is rotational invariant only when $R=a$ where a is the lattice spacing†. When $R=0$ and $\alpha_1=\beta_1$ one has the isotropic force model (or Born model). One sees that the Born model consists of a CF term with elastic constant α_1 and an external field term with coefficient β_1. We expect that turning on R or allowing the angular degrees of freedom θ_i will drive the system to a new universality class different from that of the random resistor network (RRN).

The field theory using Stephen's formalism [8] has been discussed in detail by Harris and Lubensky in [11]. The effective Hamiltonian can be calculated as

$$\mathrm{e}^{-H_{\mathrm{eff}}}=\prod_{\langle ij\rangle}\left[1-p+p\exp\left(-\sum_{\alpha}H_{ij}(\alpha)\right)\right] \tag{5}$$

or

$$-H_{\mathrm{eff}}=\sum_{\langle ij\rangle}\ln\left[1-p+p\exp\left(-\sum_{\alpha}H_{ij}(\alpha)\right)\right] \tag{6}$$

where p is the concentration. Here we have introduced n replicas to facilitate the random average and α represents the replica index. Now we decompose (6) into its Fourier components $\exp(\mathrm{i}\boldsymbol{K}_x\cdot\boldsymbol{u}_{ix})$, $\exp(\mathrm{i}\boldsymbol{K}_y\cdot\boldsymbol{u}_{iy})$ and $\exp(\mathrm{i}\boldsymbol{P}\cdot\boldsymbol{\theta}_i)$, where $\boldsymbol{K}_x$, $\boldsymbol{K}_y$ and $\boldsymbol{P}$ are n-component vectors. We obtained (up to a constant):

$$\begin{aligned}H_{\mathrm{eff}}=-\sum_{\langle ij\rangle}\sum_{\boldsymbol{K}}B_{\boldsymbol{K}}&\exp[\mathrm{i}\boldsymbol{K}_x\cdot(\boldsymbol{u}_{ix}-\boldsymbol{u}_{jx})]\exp[\mathrm{i}\boldsymbol{K}_y\cdot(\boldsymbol{u}_{iy}-\boldsymbol{u}_{jy})]\exp[-\mathrm{i}RR_{ijy}\boldsymbol{K}_x\cdot(\boldsymbol{\theta}_i+\boldsymbol{\theta}_j)\\&\times\exp[\mathrm{i}RR_{ijx}\boldsymbol{K}_y\cdot(\boldsymbol{\theta}_i+\boldsymbol{\theta}_j)]\exp[\mathrm{i}\boldsymbol{P}\cdot(\boldsymbol{\theta}_i-\boldsymbol{\theta}_j)]\end{aligned} \tag{7}$$

where

$$B_{\boldsymbol{K}}=\sum_{l=1}^{\infty}\frac{(-1)^{l+1}}{l}v^l\exp\left(-\frac{P^2}{2l\gamma_1R^2}\right)\exp\left(-\frac{K_x^2+K_y^2}{2l\alpha_1}\right) \tag{8}$$

where we have set $\alpha_1=\beta_1$ for simplicity. Here $\boldsymbol{K}_\alpha\equiv K_{x\alpha}\hat{e}_1+K_{y\alpha}\hat{e}_2+P_\alpha\hat{e}_3$ and $v=p/(1-p)$. Now we define the following quantities:

$$\begin{aligned}\Phi_0(\boldsymbol{K})&=\exp(\mathrm{i}\boldsymbol{K}_x\cdot\boldsymbol{u}_{ix})\exp(\mathrm{i}\boldsymbol{K}_y\cdot\boldsymbol{u}_{iy})\exp(\mathrm{i}\boldsymbol{P}\cdot\boldsymbol{\theta}_i)\\\Phi_{i1}^{(x)}(\boldsymbol{K})&=\Phi_0(\boldsymbol{K})\cos(R\boldsymbol{K}_y\cdot\boldsymbol{\theta}_i)\\\Phi_{i2}^{(x)}(\boldsymbol{K})&=-\Phi_0(\boldsymbol{K})\sin(R\boldsymbol{K}_y\cdot\boldsymbol{\theta}_i)\\\Phi_{i1}^{(y)}(\boldsymbol{K})&=\Phi_0(\boldsymbol{K})\cos(R\boldsymbol{K}_x\cdot\boldsymbol{\theta}_i)\\\Phi_{i2}^{(y)}(\boldsymbol{K})&=\Phi_0(\boldsymbol{K})\sin(R\boldsymbol{K}_x\cdot\boldsymbol{\theta}_i).\end{aligned} \tag{9}$$

So (7) can be written as

$$H_{\mathrm{eff}}=-\frac{1}{2}\sum_{l=x,y}\sum_{ij}\sum_{\boldsymbol{K}}B_{\boldsymbol{K}}\gamma_{ij}^{(l)}[\Phi_{i1}^{(l)}(\boldsymbol{K})\Phi_{j1}^{(l)}(-\boldsymbol{K})+\Phi_{i2}^{(l)}(\boldsymbol{K})\Phi_{j2}^{(l)}(-\boldsymbol{K})] \tag{10}$$

where $\gamma_{ij}^{(x)}$ ($\gamma_{ij}^{(y)}$) is the nearest-neighbour indicator along the x (y) direction, i.e. it is one if sites i and j are nearest neighbours along the x (y) direction and zero otherwise. Since there is no dilution for the site, we can add the kinetic energy H' of the disc system to H_{eff}, where

$$H'=-\frac{1}{2}\sum_{i\alpha}(m\omega_1^2u_{ix\alpha}^2+m\omega_1^2u_{iy\alpha}^2+\tfrac{1}{2}mR^2\omega_2^2\theta_{i\alpha}^2).$$

† As pointed out by Lubensky.

After making a Hubbard–Stratanovich transformation, we obtain

$$\mathrm{e}^{-H_{\mathrm{eff}}} = \int (\mathrm{D}\Psi) \exp\left(-\frac{1}{2}\sum_{l=x,y}\sum_{ij}\sum_{\boldsymbol{K}} B_{\boldsymbol{K}}^{-1}(\gamma_{ij}^{(l)})^{-1}(\Psi_{i1}^{(l)}(\boldsymbol{K})\Psi_{j1}^{(l)}(-\boldsymbol{K}) + \Psi_{i2}^{(l)}(\boldsymbol{K})\Psi_{j2}^{(l)}(-\boldsymbol{K}))\right) \mathrm{e}^{-H_1-H'} \tag{11}$$

where

$$H_1 = \sum_{l=x,y}\sum_{i}\sum_{\boldsymbol{K}} [\Psi_{i1}^{(l)}(\boldsymbol{K})\Phi_{i1}^{(l)}(-\boldsymbol{K}) + \Psi_{i2}^{(l)}(\boldsymbol{K})\Phi_{i2}^{(l)}(-\boldsymbol{K})] \tag{12}$$

where Ψ is a field variable.

The mean-field equation can be obtained by evaluating the Ψ integral in $\int \mathrm{D}u\, \mathrm{D}\theta\, \mathrm{e}^{-H_{\mathrm{eff}}}$ by steepest descents. Differentiating the exponent with respect to Ψ gives the mean-field equation

$$B_{\boldsymbol{K}}^{-1}(\gamma_{ij}^{(x)})^{-1}\Psi_{i1}^{(x)}(\boldsymbol{K}) = \langle\Phi_{j1}^{(x)}(\boldsymbol{K})\rangle \tag{13}$$

$$B_{\boldsymbol{K}}^{-1}(\gamma_{ij}^{(y)})^{-1}\Psi_{i1}^{(y)}(\boldsymbol{K}) = \langle\Phi_{j1}^{(y)}(\boldsymbol{K})\rangle \tag{14}$$

where $\langle A\rangle$ denotes $\int \mathrm{D}u\, \mathrm{D}\theta A\, \mathrm{e}^{-H_1-H'}/Z_1$ and $Z_1 = \int \mathrm{D}u\, \mathrm{D}\theta\, \mathrm{e}^{-H_1-H'}$. Here we have used the summation convention. We now take the Fourier transformation of (13):

$$\Psi_{i1}^{(x)}(\boldsymbol{W}) \equiv \int \mathrm{D}P\, \mathrm{D}K_x\, \mathrm{D}K_y\, \mathrm{e}^{\mathrm{i}W_xK_x}\, \mathrm{e}^{\mathrm{i}W_yK_y}\, \mathrm{e}^{\mathrm{i}W_\theta P}\Psi_{i1}^{(x)}(\boldsymbol{K}) \tag{15}$$

where $\boldsymbol{W} = W_x\hat{e}_1 + W_y\hat{e}_2 + W_\theta\hat{e}_3$. We obtain

$$\begin{aligned}
B_{\boldsymbol{W}}^{-1}(\gamma_{ij}^{(x)})^{-1}\Psi_{j1}^{(x)}(\boldsymbol{W})
&= \tfrac{1}{2}\langle\delta(W_\theta-\theta_j)\delta(W_x-u_{jx})\delta(W_y-u_{jy}-R\theta_j)\rangle \\
&\quad + \tfrac{1}{2}\langle\delta(W_\theta-\theta_j)\delta(W_x-u_{jx})\delta(W_y-u_{jy}+R\theta_j)\rangle \\
&= \frac{1}{2Z_1}\exp\bigg[-\frac{m}{2}\omega_1^2(W_x^2+W_y^2) - \frac{m}{2}\omega_2^2\left(\frac{3}{2}R^2W_\theta^2 - 2W_yRW_\theta\right) \\
&\quad + \frac{1}{2}\Psi_{j1}^{(x)}(\boldsymbol{W}) + \frac{1}{2}\Psi_{j1}^{(x)}(\boldsymbol{W} - RW_\theta\hat{e}_2) \\
&\quad + \frac{1}{2\mathrm{i}}\Psi_{j2}^{(x)}(\boldsymbol{W}) - \frac{1}{2\mathrm{i}}\Psi_{j2}^{(x)}(\boldsymbol{W} - 2RW_\theta\hat{e}_2) \\
&\quad + \frac{1}{2}\Psi_{j1}^{(y)}[\boldsymbol{W} - RW_\theta(\hat{e}_1+\hat{e}_2)] + \frac{1}{2}\Psi_{j1}^{(y)}[\boldsymbol{W} + RW_\theta(\hat{e}_1-\hat{e}_2)] \\
&\quad + \frac{1}{2\mathrm{i}}\Psi_{j2}^{(y)}[\boldsymbol{W} - RW_\theta(\hat{e}_1+\hat{e}_2)] - \frac{1}{2\mathrm{i}}\Psi_{j2}^{(y)}[\boldsymbol{W} - RW_\theta(\hat{e}_1-\hat{e}_2)]\bigg] \\
&\quad + \text{term with } R \text{ replaced by } -R
\end{aligned} \tag{16}$$

where

$$B_W = B_0 + \frac{D_3^2}{4\gamma_2R^2} + \frac{D_1^2+D_2^2}{4\alpha_2} + \ldots \tag{17}$$

and $B_0=\ln(1+v)$, $D_1^2=\partial^2/\partial W_x^2$, $D_2^2=\partial^2/\partial W_y^2$ and $D_3^2=\partial^2/\partial W_\theta^2$. We now expand (16) with respect to RW_θ and keep the quadratic term of W_θ. We obtain

$$B_W^{-1}(\gamma_{ij}^{(x)})^{-1}\Psi_{j1}^{(x)}$$
$$=\frac{1}{2Z_1}\exp\left(\Psi_{i1}^{(x)}+\Psi_{i1}^{(y)}-\frac{m}{2}\omega_1^2W^2-\frac{3m}{4}\omega_2^2R^2W_\theta^2\right)$$
$$\times(2+2R^2W_\theta^2D_2^2\Psi_{i1}^{(x)}+R^2W_\theta^2D^2\Psi_{i1}^{(y)}) \tag{18}$$

where we have written $D^2=D_1^2+D_2^2$ and $W^2=W_x^2+W_y^2$.

Similarly, from (14), we have

$$B_W^{-1}(\gamma_{ij}^{(y)})^{-1}\Psi_{j1}^{(y)}$$
$$=\frac{1}{2Z_1}\exp\left(\Psi_{i1}^{(x)}+\Psi_{i1}^{(y)}-\frac{m}{2}\omega_1^2W^2-\frac{3m}{4}\omega_2^2R^2W_\theta^2\right)$$
$$\times(2+2R^2W_\theta^2D_1^2\Psi_{i1}^{(y)}+R^2W_\theta^2D^2\Psi_{i1}^{(x)}). \tag{19}$$

Note that $\Psi_{i1}^{(x)}$ and $\Psi_{i1}^{(y)}$ are symmetric in (18) and (19); we assume that they have the same scaling form in the critical region, i.e. $\Psi_{i1}^{(x)}\sim\Psi_{i1}^{(y)}\sim S_0$. From (18) and (19), we obtain

$$S=\frac{1}{Z_1}(1+2B_WR^2W_\theta^2D^2S)\exp\left(4B_WS-\frac{m}{2}\omega_1^2W^2-\frac{3m}{2}\omega_2^2R^2W_\theta^2\right) \tag{20}$$

where $S=S_0/2B_W$ and we have used the fact that $(\gamma_{ij}^{(x)})^{-1}=(\gamma_{ij}^{(y)})^{-1}=\delta_{ij}/2$.

As discussed in detail in [8], one has $Z_1=e^{4B_0}$ to ensure the percolation threshold which is determined by $4B_0=1$. Now we look for a solution of the form

$$S=1-r_0^\beta F\left(\alpha_2W^2r_0^{\phi_1},\gamma_2W_\theta^2R^2r_0^{\phi_2},\frac{\omega_1^2}{\alpha_2r_0^{\Delta+\phi_1}},\frac{\omega_2^2}{\gamma_2R^2r_0^{\Delta+\phi_2}}\right) \tag{21}$$

where $r_0=1-4B_0$ and F is a scaling function. Substituting (17) and (21) into (20) and expanding the exponent, we have [8]

$$\tfrac{1}{2}F^2+r_0^{1-\beta}F-\frac{m\omega_1^2}{2r_0^{2\beta}}W^2-\frac{3m\omega_2^2}{4r_0^{2\beta}}R^2W_\theta^2-\frac{R^2}{r_0^\beta}W_\theta^2D^2F-\frac{1}{\alpha_2r_0^\beta}D^2F-\frac{1}{\gamma_2r_0^\beta}D_3^2F=0 \tag{22}$$

where we have omitted terms which do not contribute to (22) in the scaling region. Note that when $R=0$, we have

$$\tfrac{1}{2}F^2+r_0^{1-\beta}F-\frac{m\omega_1^2}{2r_0^{2\beta}}W^2-\frac{1}{\alpha_2r_0^\beta}D^2F=0 \tag{23}$$

which is the same form as in (4.18) of [8] with $\beta=1$. We have [8] $\phi_1=1$ and $\Delta=2$ for (23) which is the result of RRN. When R is not zero, we still have $\Delta=2$ and $\beta=1$ from (22). But we have $\phi_1=2$ and $\phi_2=1$ (a crossover). It is the term $R^2W_\theta^2r_0^{-1}D^2F$ in (22) which is responsible for the crossover.

We now consider the two-point correlation function or elastic susceptibility

$$\chi_{K_x,K_y}(x,x')=[\Psi_{K_x,K_y}(x)\Psi_{-K_x,-K_y}(x')]_{av}$$
$$=\left[\exp\left(-\frac{K^2}{2}\langle Q_1|G|Q_1\rangle\right)\right]_{av}$$
$$\simeq\chi_p(x,x')\left(1-\frac{K^2}{2\alpha_1}L^{\phi_1/\nu}\right) \tag{24}$$

where $\Psi_{K_x,K_y}(x)=\exp(\mathrm{i}K_x u_x(x)+\mathrm{i}K_y u_y(x))$, $[\ ]_{\mathrm{av}}$ denotes the random average, Q_1 is the generalised displacement for compression and G is the Green function of the system. Hence $\langle Q_1|G|Q_1\rangle^{-1}$ is a two-point effective elastic constant if we let the vector (K_x, K_y) be parallel to $x-x'$. Here $\chi_{\mathrm{p}}(x, x')$ is the susceptibility for percolation. Comparing (24) with (21), we identify ϕ_1 as the crossover exponent which describes the way $\langle Q_1|G|Q_1\rangle$ scales with the distance. Similarly we define the splay elastic susceptibility $\chi_{\mathrm{p}}(x, x')$ as:

$$\begin{aligned}\chi_{\mathrm{p}}(x, x') &= [\exp(\mathrm{i}P\theta_x)\exp(-\mathrm{i}P\theta_{x'})]_{\mathrm{av}} \\ &= \left[\exp\left(-\frac{P^2}{2}\langle Q_2|G|Q_2\rangle\right)\right]_{\mathrm{av}} \\ &\simeq \chi_{\mathrm{p}}(x, x')\left(1-\frac{P^2}{2\gamma_1}L^{\phi_2/\nu}\right)\end{aligned} \tag{25}$$

where Q_2 is the generalised displacement where the disc at x is rotated by an angle θ and the disc at x' is rotated by an angle $-\theta$ and $\langle Q_2|G|Q_2\rangle^{-1}$ is the corresponding effective elastic constant. We identify ϕ_2 as the exponent which describes the way $\langle Q_2|G|Q_2\rangle$ scales with distance. If opposite sides (of length L) of a square are displaced by u and $-u$ respectively, then the energy $E\sim 2Ku^2$. Using the node-link picture [10] the energy is that of $(L/\xi)^2$ links, each of length ξ at whose ends discs suffer a relative displacement $\sim au\xi/L$; thus we obtain

$$E\sim (L/\xi)^2(au\xi/L)^2 k\xi^{-\phi_1/\nu}. \tag{26}$$

So $K\sim k\xi^{-\phi_1/\nu}$, or $f=\phi_1$ in two dimensions. Generally in d dimensions, we then have

$$f=(d-2)\nu+\phi_1.$$

This equation gives the mean-field value $f_{\mathrm{MF}}=4$ which is consistent with the scaling theory [6]. It is not clear how to relate ϕ_1 to ϕ_2. From (21), dimensional analysis seems to suggest that $\phi_1=\phi_2+2\nu$, or $f=d\nu+\phi_2$. This equation is reminiscence of the conjectured relation [2, 11-14] $f=d\nu+\phi_{\mathrm{re}}$ where ϕ_{re} is the crossover exponent for RRN. For the bond-bending model [2, 6], we have proved [15] that $\phi_2=\phi_{\mathrm{re}}$ which supports our analysis. Although we cannot prove $f=d\nu+\phi_{\mathrm{re}}$, our result favours this conjectured relation.

In summary, we have constructed a Stephen-type mean-field theory. We have introduced two crossover exponents ϕ_1 and ϕ_2 and obtained their mean-field value. Finally, we have discussed the relation between ϕ_1, ϕ_2 and the bulk modulus exponent f.

I would like to thank Professor A B Harris for his guidance and support, and Professor T C Lubensky for helpful discussions. I thank the NSF for partial support under grant no DMR 85-19059 of the MRL program, and the NIH for support under grant no 4-60357.

References

[1] Feng S and Seng P N 1984 *Phys. Rev. Lett.* **52** 216
[2] Feng S C, Sen P N, Halperin B I and Lobb C J 1984 *Phys. Rev.* B **30** 5386
[3] Day A R, Tremblay R R and Tremblay A M S 1986 *Phys. Rev. Lett.* **56** 2501

[4] Roux S and Hansen A 1988 *Europhys. Lett.* **6** 301
Hansen A and Roux S 1988 *Universality Class of Central Force Percolation* preprint
[5] Zabolitzky J G, Bergmann D J and Stauffer D 1986 *J. Stat. Phys.* **44** 211
[6] Kantor Y and Webman I 1984 *Phys. Rev. Lett.* **52** 1891
[7] Bergman D J and Kantor Y 1984 *Phys. Rev. Lett.* **53** 511
[8] Stephen M J 1978 *Phys. Rev.* B **17** 4444
[9] Schwartz L M, Johnson D L and Feng S 1984 *Phys. Rev. Lett.* **52** 831
Feng S 1985 *Phys. Rev.* B **32** 510
[10] Skal A S and Shklovskii B I 1974 *Fiz. Tekh. Poluprovodn.* **8** 1582 (*Sov. Phys.-Semicond.* 1975 **8** 1029)
[11] Harris A B and Lubensky T C 1987 *Phys. Rev.* B **35** 6964
[12] Roux S 1986 *J. Phys. A: Math. Gen.* **19** L351
[13] Sahimi M 1986 *J. Phys. C: Solid State Phys.* **19** L79
[14] Harris A B and Lubensky T C unpublished
[15] Wang J and Harris A B 1988 *Europhys. Lett.* **6** 615
Wang J, Harris A B and Lubensky T C unpublished